

Maximum likelihood: an estimate, its precision and the likelihood-ratio test

The ideas in six lines

1. The **likelihood** is the probability of the observed data, read as a function of the parameters.
2. The **maximum likelihood estimate (MLE)** is the parameter value where the likelihood peaks.
3. For a Gaussian AR(1), given the first observation, the MLE of ρ is the **least-squares** estimate.
4. A sharper peak means more **information** and a smaller standard error. It shrinks with $1/\sqrt{T}$.
5. The **likelihood-ratio** statistic compares the peak with a candidate value. It is about χ_1^2 .
6. A long sample gives a narrow interval — only if the process stayed the same throughout.

The formulas

$$\ell(\rho, \sigma^2) = -\frac{T}{2} \ln(2\pi\sigma^2) - \frac{1}{2\sigma^2} \sum_{t=1}^T (z_t - \rho z_{t-1})^2$$

$$\hat{\rho} = \frac{\sum_t z_t z_{t-1}}{\sum_t z_{t-1}^2}, \quad \widehat{\text{se}}(\hat{\rho}) = \sqrt{\hat{\sigma}^2 / \sum_t z_{t-1}^2}$$

LR = $2[\ell(\hat{\rho}) - \ell(\rho_0)]$; reject at 5% if LR > 3.84. The 95% interval: all ρ with $\ell(\rho) \geq \ell(\hat{\rho}) - 1.92$.

Worked example

Simulated productivity series of the chapter:

- 200 quarters: $\hat{\rho} = 0.913$, standard error 0.029, 95% interval 0.86–0.97. Rules out $\rho = 1$.
- 20 quarters: 95% interval 0.70–1.10. Cannot rule out $\rho = 1$.
- From about 60 quarters on, the interval stays below 1 in this series.

Four traps

- “200 quarters give the exact value.” No. They give a precise estimate with an interval.
- “Ten times the data, ten times the precision.” No. The standard error falls with $\sqrt{T}$.
- “A long sample is always better.” Only if the process did not change within it.
- “LR tests work the same at $\rho = 1$.” Near a unit root the χ^2 approximation is unreliable.

Words to know

likelihood · log-likelihood · maximum likelihood estimate · information · standard error · likelihood-ratio test · χ^2 distribution

Try these

1. The log-likelihood is -120.4 at the estimate and -123.1 at ρ_0 . What is the LR statistic? Is ρ_0 rejected at 5%?
2. The sample size quadruples. What happens to the standard error?
3. Estimate 0.9 , standard error 0.03 . What is the Wald 95% interval? (Two decimals.)
4. Which values of ρ belong to the 95% likelihood-ratio interval?
5. For a Gaussian AR(1), the MLE of ρ equals which simple estimator?
6. Why can 20 quarters not rule out $\rho = 1$ in the chapter's series?

Answers

1. $2 \times (123.1 - 120.4) = 5.4 > 3.84$: rejected at 5%.
2. It halves.
3. $0.9 \pm 1.96 \times 0.03 = 0.9 \pm 0.06$, about $[0.84, 0.96]$.
4. Those whose log-likelihood is within 1.92 of the maximum.
5. The least-squares regression of z_t on z_{t-1} .
6. The likelihood is flat with so few observations: the interval, 0.70–1.10, includes 1.